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Primitive elements of free Lie p-algebras

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Abstract

Let L be a finitely generated free Lie p-algebra and $\langle a\rangle$ an ideal generated by $a \in L$. It is proved that $L/\langle a\rangle$ is free if and only if $\langle a\rangle$ is primitive (i.e. a belongs to some set of free generators of L). Earlier analogues theorems were proved for some objects, for example, for groups, Lie algebras, free algebras and so on.

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Introduction. It is known (1930, [4]), that if F is a finitely generated free group and $a \in F$ then a is a primitive element (i.e. a belongs to some set of free generators of F) if and only if $F/\langle a\rangle$ is a free group ($\langle a\rangle$ denotes a normal subgroup of F generated by a). Later similar theorems were proved for Lie algebras (1970, [2]), free algebras, free commutative algebras and free anticommutative algebras (2001, [6]). Mikhalev, Shpilrain and Umirbaev in (2004, [7]) conjectured that analogous theorem for Lie p-algebras is also true. In [8] the author proved Freiheitssatz for Lie p-algebras but with its help as it seems is impossible to prove the foresaid theorem. In this paper, we prove a theorem about primitive elements of free Lie p-algebras in the same manner as in (1970, [2]) using Bokuts result from [1]. Some results of our article were announced in [9].

Let k be a field of characteristic $p > 0$, $p \neq 2$, let $F = k\langle X\rangle$ be a free associative algebra without identity with $X = \{x_1, x_2, ..., x_n\}$ as a set of free generators. We will assume that $x_i < x_j \Leftrightarrow i > j$ and if w_1 and w_2 are words from $k\langle X\rangle$ then $w_1 < w_2$ either $degw_1 < degw_2$ or $degw_1 = degw_2$ and $w_1 < w_2$ lexicographically.

For $f \in F = k\langle X\rangle$, let $\bar{f}$ denote a leading word of F with nonzero coefficient. We assume that the coefficient of $\bar{f}$ is equal to one. It is clear that $\bar{f}\bar{g} = \overline{fg}$.

Let $L_p\langle X\rangle$ denotes a free Lie p-algebra over k with X as a set of free generators. A set $Y \subset L_p\langle X\rangle$ is called p-independent [2] if Y is a set of free generators of Lie p-subalgebra of $L_p\langle X\rangle$ generated by Y (recall that any Lie p-subalgebra of free Lie p-algebra is free [12]).

We recall now several definitions and results about $L_p\langle X\rangle$.

A linear basis of $L_p\langle X\rangle$ are all p-proper words [2] which are formed from symbols $\{x_1, x_2, ..., x_n\}$. If $L\langle X\rangle$ denotes a free Lie algebra free generated by the set X, then the proper words of $L_p\langle X\rangle$ are formal p^k-degrees of proper words of $L\langle X\rangle$.

We shall use the ordinary concept of degree of element from $L_p\langle X\rangle$; for example if $f = x_\alpha x_\beta + x_\gamma^p$, then $degf = p$. We assume that $deg0 = 0$.

Suppose $f \in L_p\langle X\rangle$, $f = \sum_i \alpha_i q_i$, where q_i are p-proper words. Such a record of f is called a right form of f. An element $f' = \sum_{i\in I} \alpha_i q_i$ where $degi = degf$ and $degi < degf$ if $i \notin I$ is called a major part of f. Let $\tilde{f}$ denote the major member of $f \in L_p\langle X\rangle$ defined as a lexicographically major word among q_i, $i \in I$. About these concepts see [2].

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A subset $Y \subset L_p\langle X\rangle$ is called p-reduced [3] if for any $f \in Y$ his major part f' does not belong to Lie p-subalgebra of $f \in L_p\langle X\rangle$ generated by major parts of all elements from $Y\backslash\{f\}$. We assume that the empty set is p-reduced. Let $Y = \{y_1, y_2, ..., y_m\} \subset L_p\langle X\rangle$ be a finite set. A map $t : Y \to L_p\langle X\rangle$ is called elementary if for some j

$$t(y_i) = y_i, \text{if } i \neq j,$$

$$t(y_j) = \alpha y_j + \varphi_j(y_1, y_2, ..., y_m), \text{where } y_j \text{ is missed;}$$

here $\alpha \in k$, $\alpha \neq 0$ and φ_j are polynomials i.e. elements of free Lie p-algebra with m free generators.

Let Y' denotes a set of major parts of elements from Y' with respect to standard ordering considered in the beginning this paper. Put

$$l(Y) = \sum_i deg(y_i). \tag{1}$$

As we have already noted $degy_i$ is the length of longest word in y_i and $deg0 = 0$.

Lemma 1. Let $\{y_1, y_2, ..., y_m\}$ be a finite set of generators of $L_p\langle X\rangle$. Then it exist $l(Y) - n$ (here n is a number of free generators of $L_p\langle X\rangle$) elementary maps which translate Y onto a set of generators of $L_p\langle X\rangle$ with degrees (regarding to X) less or equal one.

Remark 1. This lemma was proved in [2] for Lie algebras; we prove our lemma in the same manner.

Proof. We may assume that Y contains at least one element; otherwise there is nothing to prove. Let us prove that Y' is not p-independent. Since Y generates $L_p\langle X\rangle$ we must have

$$x_i = \sum_{i=1}^{m} \alpha_{ij} y_j + f_i(y_1, y_2, ..., y_m), \tag{2}$$

where f_i does not contain elements of degree one. Assume all f_i are zero:

$$x_i = \sum_{i=1}^{m} \alpha_{ij} y_j, i = 1, 2, ..., n. \tag{3}$$

Let us compare elements with highest degrees in (3). Assume that there exists j_0 such that $deg(y_{j_0}) > 1$, $\alpha_{i_0 j_0} \neq 0$. Then

$$(x_{i_0})' = x_{i_0} = (\sum_{j=1}^{m} \alpha_{i_0} y_j)'. \tag{4}$$

Let us denote $J = \{j | deg(y_j) = deg(y_{j_0})\}$, then from (4) follows

$$\sum_{j \in J} \alpha_{i_0} y'_j = 0, \tag{5}$$

i.e. Y' is not p-independent because otherwise we would have $deg(x_0) > 1$.

On the other hand, if in (2) we have that if $(\forall i, j)(\alpha_{ij} \neq 0 \text{ implies } deg(y_i) = 0)$, then from (3) it follows

$$x'_i = x_i = \sum_{j \in J_i} \alpha_{ij} y'_j, i = 1, 2, ..., n.$$

Any element from Y' is generated by elements x_i, therefore according to (5) all elements from Y', and among those with the degrees greater one, are generated by elements y'_j, $j \in \bigcup_i J_i$, i.e. Y' is not p-independent.

Now suppose that in (2) $f_{i_0}(y_1, y_2, ..., y_m) \neq 0$ for some i_0. If $f_{i_0}(y'_1, y'_2, ..., y'_m) = 0$ then Y' is not p-independent. Now suppose $f_{i_0}(y'_1, y'_2, ..., y'_m) \neq 0$. Let us write it as

$$f_{i_0}(y'_1, y'_2, ..., y'_m) = \sum_{j=1}^{s} h_j(x_1, x_2, ..., x_n), \tag{6}$$

where h_j is a homogeneous component of degree d_i of $f_{i_0}(y'_1, y'_2, ..., y'_m)$, $d_1 < d_2 < ... < d_s$. Because y'_i are homogeneous, each polynomial $h_j(x_1, x_2, ..., x_n)$ must be a polynomial of arguments $y'_1, y'_2, ..., y'_m$:

$$h_j(x_1, x_2, ..., x_n) = q_j(y'_1, y'_2, ..., y'_m).$$

Therefore from (6) follows

$$f_{i_0}(y'_1, y'_2, ..., y'_m) = \sum j = 1^s q_j(y'_1, y'_2, ..., y'_m),$$

where $q_j(y'_1, y'_2, ..., y'_m) \neq 0$, $j = 1, 2, ..., s$, otherwise Y' would have not been p-independent; in particular $q_s(y'_1, y'_2, ..., y'_m) = 0$. Consequently

$$(f_{i_0}(y'_1, y'_2, ..., y'_m))' = f_{i_0}(y'_1, y'_2, ..., y'_m) = q_s(y'_1, y'_2, ..., y'_m).$$

From (2) follows

$$x_i = x'_i = (\sum_{j=1}^{s} \alpha_{i_0 j} y_j + f_{i_0}(y_1, y_2, ..., y_m))'. \tag{7}$$

Two cases are now possible.

1.$f_{i_0}(y'_1, y'_2, ..., y'_m) = q_s(y'_1, y'_2, ..., y'_m)$ is contained in the major part of $\sum_{j=1}^{m} \alpha_{i_0 j} y_j$; then because the degree of x_i is one, for some $J \subset \{1, 2, ..., m\}$ we must have (see (7)):

$$\sum_{j \in J}^{m} \alpha_{i_0 j} y_j + q_s(y'_1, y'_2, ..., y'_m) = 0.$$

i.e.Y' is not p-independent.

2. $f_{i_0}(y'_1, y'_2, ..., y'_m) = q_s(y'_1, y'_2, ..., y'_m)$ is not contained in the major part of $\sum_{j=1}^{m} \alpha_{i_0 j} y_j$; then $\sum_{j=1}^{m} \alpha_{i_0 j} y_j$ contains letters y_j such that their degree are greater than d_s and consequently, greater than one. Let y_j, $j \in J$ be all y_j from $\sum_{j=1}^{m} \alpha_{i_0 j} y_j$ (of course with nonzero coefficients) having the highest degree; then

$$\sum_{j=1}^{m} \alpha_{i_0 j} y_j = 0$$

because $deg(x_{i_0} = 1$ (see (7), i.e. Y' is not p-independent. So we have considered all cases and have proved that Y' is not p-independent. In ([2], Lemma 2) was proved that a p-reduced subset of free Lie p-algebra is p-independent. From the above lemma follows, because Y' is not p-independent, that Y' is not p-reduced . Therefore there exists an element $y'_{j_0} \in (Y')' = Y'$ such that y'_{j_0} is contained in a p-subalgebra of $L_p\langle X\rangle$ generated by a set $Y' \setminus \{y'_{j_0}\}$ i.e.

$$y'_{j_0} = q(y'_1, ..., \hat{y}', ..., y'_m), \tag{8}$$

where $q(y'_1, ..., \hat{y}', ..., y'_m)$ does not contain y'_{j_0}. Consequently a map

$$y_i^{(1)} = y_i, i \neq j_0, y_{j_0}^{(1)} = y_{j_0} - q(y'_1, ..., \hat{y}', ..., y'_m) \tag{9}$$

reduces $l(Y)$ (see (1)). Indeed from (8) follows that $q(y_1, ..., \hat{y}', ..., y_m) \neq 0$ and therefore

$$y_{j_0} = q(y'_1, ..., \hat{y}', ..., y'_m) = q(y_1, ..., \hat{y}, ..., y_m)',$$

i.e. (9) reduces $l(Y)$.

Lemma 2. Assume a set $Z = \{z_1, z_2, ..., z_m\}$ generates $L\langle X\rangle$ and $deg_X z_i \leq 1$. If $\{z_1, z_2, ..., z_{m_0}\}$ is a maximal linearly independent subset of Z, then there exist $m - m_0$ elementary maps which transform Z onto the set $\{z_1, z_2, ..., z_{m_0}, 0, ...0\}$.

Proof. Let $z_j = \sum_{j=1}^{m_0} \alpha_{ij} z_j, i = m_0 + 1, m_0 + 2, ..., m$. Then it is clear that the sought maps are

$$\tilde{z}_i = z_i, i = 1, 2, ..., m,$$

$$\tilde{z}_i = z_i - \sum_{j=1}^{m_0} \alpha_{ij} z_j, i = m_0 + 1, m_0 + 2, ..., m.$$

Recall that $F = k\langle X\rangle$ is the free associative algebra over set $X = \{x - 1, x_2, ..., x_n\}$ without iden- tity (of course $X \subset F$). For $a \in F$, let $\langle a\rangle$ be an ideal of F generated by a. It is clear that $a \in \langle a\rangle$. Let $\bar{a}$ be the major word of a.

Lemma 3. If $a, b \in k\langle X\rangle$ and $\langle a\rangle = \langle b\rangle$, then a and b are linearly dependent.

Proof. If either $\langle a\rangle$ or $\langle b\rangle$ are zero, our proposition of course is valid. So we may assume that$a, b \neq 0$. From [1] follows that if $x \in \langle a\rangle$, then $\bar{a}$ is a subword of $\bar{x}$. Therefore $\bar{a}$ is a subword of $\bar{b}$ and, conversely, $\bar{b}$ is a subword of $\bar{a}$ and consequently $\bar{a} = \bar{b}$. Suppose

$$a = \alpha\bar{a} + ..., b = \beta\bar{b}; \alpha, \beta \in k, \alpha, \beta \neq 0.$$

Consider the element $c = a - \frac{\alpha}{\beta} b \in \langle a\rangle = \langle b\rangle$. If $c \neq 0$, then $\bar{c}$ is less then $\bar{a}$. On the other hand, $\bar{a}$ is a subword of $\bar{c}$ - contradiction, so $c = 0$.

Corollary 1. Let $F_1 = k\langle X\rangle_1$ be a free associative algebra with identity which is freely generated by X. Suppose $a, b \in F_1$ and $\langle a\rangle = \langle = b\rangle$. Then a and b are linearly dependent.

Proof. This is clear since $\langle a\rangle$ is an ideal in $F = k\langle X\rangle$ if and only if $\langle a\rangle$ is the ideal in $F_1 = k\langle X\rangle_1$.

Let $\langle a\rangle$ denote an ideal of $L_p\langle X\rangle$ generated by $a \in L_p\langle X\rangle$ (we assume $a \in \langle a\rangle$) and let $\bar{a}$ be the major word of a.

Corollary 2. Let $\langle a\rangle = \langle b\rangle \subseteq L_p\langle X\rangle$. Then a and b are linearly dependent.

Proof. As is well known, $u(L_p(X)) = k\langle X\rangle_1 = F_1$ (here $u(L_p(X))$ is a restricted universal enveloping algebra of $L_p(X)$). Let $\langle a\rangle$ and $\langle b\rangle$ be the ideals in $F_1 = k\langle X\rangle_1$, generated, respectively by a and b. It is clear that

$$\langle a\rangle_1 = \langle b\rangle_1 \subseteq F\langle X\rangle,$$

then according to Lemma 3 the elements a and b are linearly dependent.

Definition. An element $a \in L_p\langle X\rangle$ is primitive if there exist a set Y of free generators of $L_p\langle X\rangle$ such that $a \in Y$.

Theorem. $L_p\langle X\rangle/\langle a\rangle$ is free if and only if a is primitive in $L_p\langle X\rangle$.

Proof. It is clear that if a is primitive then $L_p\langle X\rangle/\langle a\rangle$. Suppose that $L_p\langle X\rangle/\langle a\rangle$ is free and let us prove that a is primitive.

Let us denote $\bar{L} = L_p\langle X\rangle/\langle a\rangle$. It is clear that $dim\bar{L}/\bar{L}^2 \geq n-1$. Indeed

$$\bar{L}/\bar{L}^2 = (L_p\langle X\rangle/\langle a\rangle)/(L_p\langle X\rangle/\langle a\rangle)^2 \cong L_p\langle X\rangle/(L_p\langle X\rangle^2 + \langle a\rangle);$$

but last term as k-vector space is isomorphic to $(kx_1 + kx_2 + ... + kx_n + \langle a\rangle)/\langle a\rangle$, which implies that $dim(\bar{L}/\bar{L}^2) \geq n-1$.

On the other hand, $L_p\langle X\rangle$ is generalized nilpotent, i.e. intersection all its degrees is zero. According to [5] all generalized nilpotent algebras are Hopf type, i.e. they are not isomorphic to their proper factor- algebras. Consequently,

$$rank\bar{L} = rank(L_p\langle X\rangle/\langle a\rangle) \leq n-1.$$

However,if $rank(\bar{L}) \leq n-1$, then $rank(\bar{L}/\bar{L}^2) < n-1$; so $rank(\bar{L}) = n-1$ and there exist a set of free generators $Y = \{y_1, y_2, ..., y_{n-1}\}$ for $\bar{L}$. The set $X = \{x_1, x_2, ..., x_n\}$ generates $\bar{L}$ and by to Lemma 1 there exist elementary maps which transform $\bar{X}$ in a set of generators $Z = \{z_1, z_2, ..., z_r\}$ of $\bar{L}$ such that degrees of $z_i, i = 1, 2, ..., r$ with respect Y are not greater than one. By lemma 2 there exist elementary maps which transform $Z = \{z_1, z_2, ..., z_r\}$ onto $\{z_1, z_2, ..., z_{r_0}, 0, ..., 0\}$, where $\{z_1, z_2, ..., z_{r_0}\}$ is a maximal linearly independent set in L. It is clear that $r_0 = n-1$ and number of zeros in $\{z_1, z_2, ..., z_{r_0}, 0, ..., 0\}$ is one, therefore some elementary maps transform $\{z_1, z_2, ..., z_{r_0}, 0\}$ on $\{y_1, y_2, ..., y_{r_0}, 0\}$ (if this set contains only zero then $n = 1$). Therefore we may assume that there exist elementary maps $\varphi_1, \varphi_2, ..., \varphi_s$ which transform $\bar{X} = \{\bar{x_1}, \bar{x}_2, ..., \bar{x}_n\}$ onto $\{y_1, y_2, ..., y_{r_0}, 0\}$. The elements $X = \{x_1, x_2, ..., x_n\}$ are preimages of $\bar{X} = \{\bar{x_1}, \bar{x}_2, ..., \bar{x}_n\}$. The maps $\varphi_1, \varphi_2, ..., \varphi_s$ transform X on a set $\{t_1, t_2, ..., t_n\}$ of free generators of $L_p\langle X\rangle$. Let us consider a projection $\pi : L_p\langle X\rangle \to L_p\langle X\rangle/\langle a\rangle$. From a commutative diagram below it is clear that $\pi(t_n) = 0$:

$$\begin{array}{ccc} \{x_1, x_2, ..., x_n\} & \xrightarrow{\pi} & \{\bar{x}_1, \bar{x}_2, ..., \bar{x_n}\} \\ \downarrow & & \downarrow \\ \{t_1, t_2, ..., t_n\} & \xrightarrow{\pi} & \{y_1, y_2, ..., y_{n-1}, 0\} \end{array},$$

where vertical maps are equal to composition φ of the maps $\varphi_1, \varphi_2, ..., \varphi_s$. So $t_n \in \langle a\rangle$, i.e. $\langle t_n\rangle \in \langle a\rangle$. In fact, $\langle t_n\rangle = \langle a\rangle$. Indeed let us consider an algebra $L_p\langle X\rangle/\langle t_n\rangle$. It is free. As $\langle t_n\rangle \in \langle a\rangle$ so

$$(L_p\langle X\rangle/\langle t_n\rangle)/(\langle a\rangle/\langle t_n\rangle) \cong L_p\langle X\rangle/\langle a\rangle.$$

As $L_p\langle X\rangle/\langle t_n\rangle$ and $L_p\langle X\rangle/\langle a\rangle$ are free Lie p-algebras with $n-1$ generators, and free Lie p-algebras are Hopf type algebras we must have $\langle a\rangle/\langle t_n\rangle = 0$, i.e. $\langle a\rangle = \langle t_n\rangle$. Then from Corollary 2 follows that $a = \alpha t_n$ for some $\alpha \in k$, i.e. a is primitive.

Remark 2. We assume that the other results from [2] can be proved in the same way.

Remark 3. J. P. Serre has proved the following theorem ([11], [10]):

Theorem (Serre). Let R be a commutative ring and let G be a group having no R-torsion. If H is a subgroup of finite index in G, then $cd_R G = cd_R H$.

We assume that an analogous statement about Lie p-algebras is also valid: let L be a Lie p-algebra such that restricted universal algebras of all finite Lie p-subalgebras of L are semisimple. If H is a Lie p-subalgebra of finite index in L, then $cd_L G = cd_R H$.

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